Homework 4
Math 117 - Summer 2022

1) Consider the real vector space $V = \mathbb{R}^n$, and let $\gamma \in V^*$. The point of this problem is to get another interpretation of the dual space in this case.

We mentioned in class that we can "think" of γ as a "row" vector by considering its matrix representation under the standard basis (it will be a $1 \times n$ matrix, ie a row vector.) Let us now prove that every row vector gives rise to a linear functional. Let $(x_1, \ldots, x_n)$ be a "row" vector (ie, a $1 \times n$ matrix) with each $x_i \in \mathbb{R}$ and define the functional $\varphi : V \to \mathbb{R}$ by

$$\varphi(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}) := (x_1, \ldots, x_n)(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}) = x_1a_1 + x_2a_2 + \ldots x_na_n$$

First off, convince yourself that this is actually a linear functional

(a) (2 points) Recall that there is a bijection $V^* = \mathcal{L}(V, \mathbb{R}) \simeq M_{1\times n}(\mathbb{R})$ that sends a linear map to its matrix representation. Show that this construction above (that sends a $1 \times n$ matrix to the linear functional φ) is just the inverse of this isomorphism. (In other words, if you start with a linear functional, take its matrix, and then define this new linear functional as above, you get the original linear functional you started with)

(b) (1 point) Under this identification above, and letting $V = \mathbb{R}^3$ what is the row vector that corresponds to the dual basis vector e_1^*? How about e_3^*?

Solution:

2) (2 points) Let V, W be vector spaces over $\mathbb{F}$ and let $T : V \to W$ be a linear map. Prove that if T is injective, then V is isomorphic to a subspace of W.

Solution:

3) Suppose V is an n-dimensional $\mathbb{F}$ vector space and let $T : V \to V$ be a linear map.

(a) (3 points) Suppose that T is an isomorphism, and let T^{-1} denote its inverse. Using our definition of determinant of T, prove that $det(T^{-1}) = (det(T))^{-1}$ (Hint: what is the determinant of the identity map?)

(b) (3 points) Again using our definition of determinant, Show that T is an isomorphism $\iff det(T) \neq 0$. (Hint: for one direction use part a. For the other direction it may help to use some results we proved in hws about linear maps between vector spaces of the same dimension ...)

Solution:

4) (3 points) Let V be an $\mathbb{F}$ vector space of dimension n. Prove that, for $k \leq n$ the vectors $v_1, v_2, \ldots, v_k$ are linearly independent in V $\iff v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$ in $\bigwedge^k(V)$ (Hint: extend basis....)

Solution:

5) Let V be 3 dimensional with basis $\mathcal{B} = (e_1, e_2, e_3)$, and let $\varphi : V \to V$ be a linear map

(a) (3 points) Prove that there exists a linear map $Tr(\varphi) : \bigwedge^3(V) \to \bigwedge^3(V)$ that sends the simple tensor

$$Tr(\varphi)(e_1 \wedge e_2 \wedge e_3) = \varphi(e_1) \wedge e_2 \wedge e_3 + e_1 \wedge \varphi(e_2) \wedge e_3 + e_1 \wedge e_2 \wedge \varphi(e_3)$$

We call this map the trace map of φ, and we call the resulting number that scales the basis vector $e_1 \wedge e_2 \wedge e_3$ the trace of φ

(b) (3 points) Consider the case $V = \mathbb{R}^3$ with standard basis. View the 3x3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

as a linear map and verify that our definition of trace lines up with the usual one.

Solution:

Remark: There is of course nothing special about the $n = 3$ case: One can define the trace operator for an n-dimensional vector space just as easily, as the following sum:

$$Tr(\varphi)(e_1 \wedge e_2 \wedge \cdots \wedge e_n) := \sum_{i=1}^{n} e_1 \wedge \cdots \wedge \varphi(e_i) \wedge \cdots \wedge e_n$$

Then one can pretty easily show that all the usual properties of trace hold using this definition:

NOT FOR CREDIT, JUST FOR YOUR OWN AMUSEMENT: Show the following: For $\varphi_1, \varphi_2 : V \to V$ linear maps from V to V we have

(a) $\mathrm{Tr}(\varphi_1 + \varphi_2) = Tr(\varphi_1) + Tr(\varphi_2)$

(b) $Tr(\varphi_1 \circ \varphi_2) = Tr(\varphi_2 \circ \varphi_1)$

(c) Convince yourself that (b) proves that the Trace is independent from choice of basis

(d) Is it true that $Tr(\varphi_1 \circ \varphi_2) = Tr(\varphi_1)Tr(\varphi_2)$?

6) (3 points) Let V be an inner product space (over $\mathbb{C}$ or $\mathbb{R}$), with an orthonormal basis $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ and let $T : V \to V$ be linear. Prove that

$$[T]_{\mathcal{B}} = \overline{[T^*]}_{\mathcal{B}}^{tr}$$

(That is, the matrix of the adjoint of T is the conjugate transpose of the matrix of T)

Solution: